ON THE STABILITY OF A SLEEPING TOP.

Abstract of a Lecture before the American Mathematical Society at the Princeton Meeting, October 17, 1896.

BY PROFESSOR FELIX KLEIN.

In the four lectures* of the earlier part of the week I have attempted to simplify the formulæ for the motion of a top by turning to account the methods of the modern theory of functions. In treating this problem I have been largely influenced by the consideration that it is desirable on both sides to reinforce the relationships between pure mathematics and mechanics.

To-day I consider from the same standpoint a much more elementary question, which, however, for this very reason serves as a type for many related problems, viz., the stability of a top rotating about an axis directed vertically upward. The point of support we will assume to be fixed. If it were moveable in a horizontal plane, the formulæ would be somewhat more complicated, but the final result would be quite similar to that in the special case.

When the rotation is very rapid the behavior of the top is as if its axis were held fixed by a special force. This idea was employed, for instance, by Foucault (1851); to regard it, however, as an independent mechanical principle, as is done in many presentations of the subject, is, of course, absurd.

The usual mode of attacking the problem is by means of the *method of small oscillations*. If x, y are the horizontal coördinates of the point of support of the top, n its rotational velocity, and P the moment of its weight, then, rejecting higher powers of x and y, we obtain the linear homogeneous differential equations with constant coefficients

$$x'' + ny' - Px = 0,$$
$$y'' - nx' - Py = 0.$$

The terms in x' and y' in these equations are known as the gyroscopic terms. The solutions of the equations involve the characteristic exponent

$$\lambda = \frac{\pm in \pm \sqrt{4P - n^2}}{2}.$$

* Four lectures "On the theory of the top," delivered at the invitation of Princeton University in connection with its sesquicentennial celebration.

With respect to the form of this exponent two cases are customarily distinguished: the *stable* case, $n^2 > 4P$, and the *unstable* case $n^2 \leqq 4P$, the conclusion then being drawn that in the former case actual oscillations take place about the position of equilibrium, while in the latter case the axis moves away indefinitely from the position of equilibrium.

For the stable case we obtain

$$x = a\cos\frac{nt}{2}\cdot\sin\sqrt{\frac{n^2-4P}{4}}\,t,$$

$$y = a\sin\frac{nt}{2}\cdot\sin\sqrt{\frac{n^2-4P}{4}}\,t,$$

where a is a constant of integration.

I will retain the designations "stable" and "unstable" for the cases $n^2 > 4P$ and $n^2 \leqq 4P$, and will then examine whether the motion actually corresponds to the common use of these terms.

From the start this method of small oscillations lies open to severe criticism. In the so-called unstable case it is directly self-contradictory, since the quantities, which in the construction of the differential equation are assumed to be *small*, become after its integration *large*. There is no reason whatever, therefore, for regarding the results as an approximation to the actual conditions. Even in the stable case the method lacks an accurate basis.

Poincaré, in the corresponding questions of astronomy, carries out the development in series to higher terms. But, supposing that these series converge at all, will their region of convergence extend far enough so that the actual character of the motion can be deduced from them? In the case of the top we are relieved of the laborious investigation of this question, inasmuch as the complete integration can be carried out in explicit form.

I propose the following mode of treating the problem. For the sake of simplicity, the moments of inertia of the top about its principal axes are all assumed equal to 1. The axis, being originally vertical, let the polar angles at any time to be ϑ, ψ, and let $\cos\vartheta = u$. The formulæ of integration are then

$$t = \int\frac{du}{\sqrt{U}}, \qquad \psi = n\int\frac{du}{(u+1)\sqrt{U}},$$

where $$U = 2(u-1)\Big(n^2 + (Pu-2)(u+1)\Big).$$

The upper end of the axis (apex) of the top describes in all cases on the surface of the circumscribed sphere a rosette consisting of a number of congruent loops. This is still the case when $n=0$, a loop being then identical with a great circle of the sphere. Our interest centres in the question, how *long* these loops are, *i. e.*, to what value $u=e$ does u diminish, beginning with $u=1$. Here $u=e$ is that root of $U=0$ which lies between $u=+1$ and $u=-1$. In order to obtain the width of the loops it would be necessary to discuss the integral ψ.

Introducing v to denote the value when $u=1$ of the angular velocity $d\vartheta/dt$ of the axis of the top, this being equal to the measure of the lateral impulse by which the axis is carried out of the vertical position, we have from $U=0$, on writing e for u,

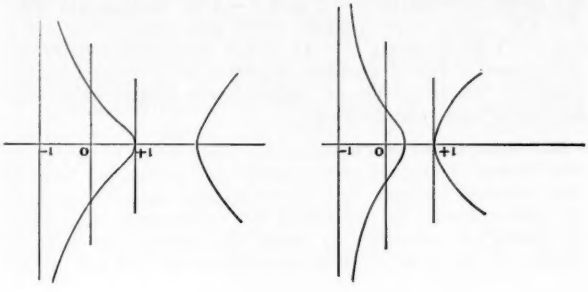
$$v^2 = \frac{(1-e)\,(n^2-2P)\,(e+1)}{e+1}.$$

When e and u are rectangular coördinates, this equation properly interpreted, represents a plane cubic, symmetric to the axis of e, with a vertical tangent at $e=1$, $v=0$, and having $e+1=0$ as an asymptote. This curve has a certain difference of position according as

$$n^2-4P>0 \text{ or } n^2-4P<0,$$

(the case $n^2-4P=0$ may be disregarded for the sake of brevity). In the former (stable) case, the odd branch of the curve passes through $e=+1$, $v=0$, while in the latter (unstable) case, it is the even branch which passes through this point.

In both cases it is the odd branch which is of account for the real motion of the top since $u=\cos\vartheta$ lies, for real ϑ, between -1 and $+1$. In both cases, too, the difference $1-e$, *i.e.*, the length of the loops of the rosette, diminishes with r.



The characteristic distinction between the two cases is this: that for $n^2 - 4P > 0$, the difference $1 - e$ diminishes with v to 0, while if $n^2 - 4P < 0$ this difference never passes a certain lower limit different from 0. Accordingly, in the unstable case, the loops of the rosette take at once a certain finite length even for the smallest lateral impulse given the top.

Theoretically, this furnishes a sharp distinction between the two cases; practically, however, this may become unnoticeable, if $n^2 - 4P$ while < 0, becomes very small in absolute value. The rosette in the unstable case can become as small as we please; and given a stable rosette, a proper choice of the constants n and v will give for the unstable case a rosette *smaller* than the stable one.

Our result is therefore discordant with the common acceptation of the terms "stable" and "unstable." Besides that it does not substantiate the pretensions of the method of small oscillations. If the apex of the top in an unstable case describes a "small" rosette, why does not this fact appear from the method of small oscillations?

The answer to this last question will be apparent, if we introduce the quantity e in the integral t:

$$t = \int \frac{du}{\sqrt{\frac{2(u-e)(1-u)}{1+e}\left(n^2 - 4P - P(u-1)(e-1) + 2(u-1) + 2(e-1)\right)}}.$$

The method of small oscillations neglects in the parenthesis

$$n^2 - 4P - P(u-1)(e-1) + 2(u-1) + 2(e-1)$$

the terms containing $u - 1$ and $e - 1$ in comparison with $n^2 - 4P$. This is admissible when and only when $u - 1$ and $e - 1$ being small, $n^2 - 4P$ *is not small*,—and therefore those cases, stable or unstable, where $n^2 - 4P$ is itself a small quantity are incapable of approximate treatment by the method of small oscillations.

PRINCETON, *October* 18, 1896.

BIBLIOGRAPHY OF SURFACES AND TWISTED CURVES.

BY DR. J. E. HILL.

THE following article gives a brief sketch of a bibliography of surfaces and twisted curves, prepared by the writer, and contains some miscellaneous notes pertaining to the general subject. It comprises for the most part extracts from a paper read before the AMERICAN MATHEMATICAL SOCIETY at its last May meeting.

The bibliography is intended to represent a compilation and classification, as complete as possible (with certain exceptions), of all articles upon surfaces and twisted curves published during the present century. The exceptions noted above are the general discussion of quadric surfaces and sphero-conics and the complete consideration of line geometry. References to papers upon the curvature, lines of curvature, geodesics, umbilics, and general surface curves of quadric surfaces have been included and classified; but, otherwise, articles upon the subjects whose exclusion has been mentioned are listed only incidentally.

The bibliography is indexed, subject to revision, as follows: 1. Surfaces in general. 2. Curvature of surfaces; (a) of quadrics, (b) in general. 3. Lines of curvature; (a) upon quadrics, (b) in general. 4. Geodesics; (a) upon quadrics, (b) in general. 5. Surfaces curves; (a) upon quadrics, (b) in general. 6. Umbilics. 7. Contact of surfaces. 8. Curvilinear coördinates. 9. Algebraic surfaces. 10. Cubic surfaces. 11. Quartic surfaces; (a) in general, (b) the wave surface, (c) the torus, (d) the cyclide, (e) Kummer's surface, (f) Steiner's surface, (g) the surface of elasticity. 12. Surfaces of the fifth and higher orders; (a) quintic surfaces, (b) sextic surfaces, (c) septic surfaces, (d) octic surfaces, (e) 9-thic surfaces, (f) 10-thic surfaces, (g) 12-thic surfaces. 13. Surfaces of revolution. 14. Orthogonal surfaces. 15. Envelopes. 16. Reciprocal surfaces. 17. Minimal surfaces. 18. Ruled surfaces; (a) in general, (b) scrolls, (c) torses. 19. Surfaces defined by curvature; (a) surfaces of constant curvature, (b) surfaces with plane lines of curvature, (c) surfaces with spherical lines of curvature, (d) other special forms. 20. Special forms of surfaces; (a) applicable, (b) helicoidal, (c) isothermal, (d) parallel, (e) normal, (f) polar, (g) pseudo, (h) pedal, (j) cyclic, (k) flexible and inextensible, (l) derived, (m) focal,

(n) anallagmatic, (o) apsidal, (p) inverse, (q) other unclassified forms. 21. Twisted curves in general. 22. Curvature of twisted curves. 23. Algebraic twisted curves. 24. Twisted cubics. 25. Twisted quartics; (a) in general, (b) of the first kind, (c) of the second kind. 26. Twisted curves of the fifth and higher orders; (a) twisted quintics, (b) twisted sextics, (c) twisted septics, (d) twisted octics, (e) twisted 10-thics. 27. Other special, unclassified forms of twisted curves.

The bibliography, in its present form, consists of 3,715 different references. Each reference gives the complete title of the paper listed, the name of the author, the year and the journal of publication, the number of the journal and the number of the pages occupied by the article. At the end of each section, cross-references to articles of other sections are given.

Section 1, devoted to surfaces in general, contains 489 references, besides cross-references to some sixty other articles; the dates of publication of the papers referred to extend from 1784 to 1896. In order that this section, containing such a large number of titles, should be readily accessible to reference, an index has been appended to the section, enumerating the following divisions: Generation, Intersection, Deformation, Quadrature, Singularities and Transformation (including Representation, Projection and Correspondence). All articles of this section not included in these divisions group themselves under the very convenient head of General Theory.

It is interesting to note the names of the mathematicians who, during the first half of this century, concerned themselves with the subjects comprised in this section. Chief among such may be mentioned Monge, Lamé, Crelle, Gauss, Binet, Specht, Olivier, Gudermann, Plücker, Möbius, Tortalini, Löwestern, Gregory, Cauchy, Steiner, Bedetti, Bertrand, Bonnet, Umpfenbach, Ellis, Amiot, Schläfli, Bouquet, Salmon, Bravais, Joachimsthal, Lebesgue, Hachette, Gergonne, Fregier, Bobillier, Hamilton, Conti, Dirksen. Brunn, Boole, and Chelini; some of these are the names of men already forgotten. The first reference in this section is that to a fourteen page article, by Monge, "Sur l'expression analytique de la génération des surfaces courbes," published in the first volume of the *Mémoires de l'Académie des Sciences de Turin*, 1784.

The first article, listed in section 2, is a six-page paper, by Hachette, published in 1804 in the *Correspondance sur l'École Polytechnique* and entitled "De quelques pro-

priétés des rayons de courbure d'une surface." This section contains 193 references and is introduced by references to articles by Hachette, Binet, Gergonne, Rodrigues, Sturm, Schubert, Plücker, Sophie Germaine, Poisson, Minding, Transon, Fink, Chelini, Paignon, Thomson, Joachimsthal, Babinet, Liouville, Bonnet, Breton, Dienger and Lebesque. The majority of these names, it will be noted, are French; in fact, in this section, as well as in section 1, nearly every article referred to, appearing during the first quarter of this century, was written by some French mathematician. As a matter of fact, Meusnier read his celebrated "Mémoire sur la courbure des surfaces" before the French Academy of Sciences in February, 1776, and for the next fifty years this subject was an enthusiastic one for the French school. The first reference to the curvature of quadrics, specifically treated, is that to a paper by Dupin, published in 1804 in the last-mentioned journal; it is entitled "Détermination des rayons de courbure des surfaces du second degré."

In referring to the lines of curvature of surfaces, the first article noted is the 20-page paper of Monge in the *Journal de l' École Polytechnique*, 1795, entitled "Sur les lignes de courbure de la surface de l'ellipsoide." The first English article listed is that of Lardner, in the *Transactions of the Irish Academy*, 1824. volume 14, being "An investigation of the lines of curvature of ellipsoids, hyperboloids and paraboloids." The English mathematicians early devoted themselves to the study of this branch of the theory of surfaces, the names of Ellis, Cayley, and Thomson frequently appearing, while the name of Michael Roberts is one inseparably connected with the history and study of the lines of curvature of surfaces. There are 78 references in this section of the bibliography.

The study of geodesics is, of course, most closely connected with that of the general line of curvature and most articles upon the one deal with the other. The first article, however, that I note as treating specifically of geodesics is Euler's "Accuratior evolutio problematis de linea brevissima in superficie quacunque ducenda," first published in the *Nova Acta Academiæ Scientiarum Petropolitanæ*, 1806, volume 15. Of the English mathematicians, James Ivory appears to have been the first actually to contribute to the theory of geodesics, and his article in the 67th volume of the *Philosophical Magazine*, "On the properties of the lines of shortest distance traced on the surface of an oblate spheroid," is the first article of the century expressedly investigating this subject from the standpoint of the quadric sur-

face. The papers of Jacobi, Joachimsthal, Liouville, Chasles, Gauss and Michael Roberts are to be especially mentioned. What was written of this last distinguished mathematician in the last paragraph, holds rigorously true under this. The bibliography contains 135 direct references to articles upon geodesics.

The general surface curve is listed independent of any relation it may bear to the curvature of the surface and has, on this account, an inseparable relation to the intersection of surfaces and the general twisted curve; I have placed it in a section by itself, however, because I deemed that the number of cases in which it has been treated, *in ipso*, demanded that I should do so. There are 104 references to this section and the first reference is to a paper by Legendre, "Analyse des triangles tracés sur la surface d'un spheroid," published in the sixth volume of the *Mémoires de la classe des sciences mathématiques et physiques de l'Institut*, 1806.

I am able to find but twenty papers treating specifically of umbilics, the first reference being to the paper of Monge and Hachette, "Application d'algebra à la géomètrie," in the fourth volume of the *Journal de l'École Polytechnique*, 1802. While this paper does not actually treat of umbilics, it deserves its place at the head of the list of papers upon this subject in that it is the first paper in which the discussion of the conditions for the circular sections of quadrics is undertaken and the idea of umbilics broached. Although there are but twenty references, some of the papers referred to are worthy of much consideration, *e. g.*, those by Verdam, 1852 in the fifth volume of the *Tijdschrift voor de Wis- en Natuurkundige Wettenschappen;* by Voss, 1875, in the ninth volume of the *Mathematische Annalen;* and by Servais, 1885, in the eighteenth volume of the third series of the *Bulletin de l'Académie des sciences, des lettres et des beaux-arts de Belgique.*

The first paper treating manifestedly of the contact of surfaces is Hachette's "De la courbe de contact d'une surface conique avec une surface dont l'equation est du degré *m*," in the first volume of the *Correspondance sur l'École Polytechnique*, 1804. Of the twenty-six references listed in this section, those to the papers of Spottiswoode, extending through the '70's, are especially noticeable.

Gauss first introduced the general theory of curvilinear coördinates in his celebrated memoir, "Disquisitiones circa superficies curvas," published in 1823 and translated into French in the ninth volume of the first series of the *Nouvelles Annales de Mathématiques*, 1852. The first English

paper that I find is Gregory's "On the method of spherical coördinates," published in the first and second volumes of the *Cambridge Mathematical Journal*, 1839. This section contains thirty-nine references; and the well-known paper of Lamé, "Léçons sur les coördonnées curvilignes," Paris, 1859, and the lengthy memoir of de Salvert in the *Annales de la Société scientifique de Bruxelles*, 1884, are particularly noticeable.

Section 9, upon algebraic surfaces, which is introduced to abbreviate section 1 and fittingly to prepare the way to references to articles upon surfaces of determinate orders, is headed by Lardner's, "On the general properties of algebraic surfaces," published in the 14th volume of the *Transactions of the Irish Academy*, followed by Plücker's historic paper, "Récherches sur les surfaces algébriques des tous les degrés," appearing in the 19th volume of the *Annales de Mathématiques pures et appliquées* (Gergonne). There are 271 references in this section.

It is remarkable that the first paper that I can find that deals specifically with the cubic surface is one by L. Mossbrugger, "Untersuchungen über die geometrische Bedeutung der constanten Coefficienten in den allgemeinen Gleichungen der Flächen des zweiten und dritten Grades." This paper appeared in the first volume of the *Archiv der Mathematik und Physik*, 1841. Salmon's and Cayley's papers upon the triple tangent planes of surfaces of the third order, 1849, published in the fourth volume of the *Cambridge and Dublin Mathematical Journal*, are the first English papers listed in this section. Although one may say that the classification of cubic surfaces is practically complete, the study of these surfaces appears, still to-day, to have the same fascination as was exhibited when the discovery of the existence and the relations of the 27 lines of the general cubic surface was first announced, no less than a round dozen of worthy articles upon cubic surfaces appearing last year. Of the 205 articles referred to in this section, nearly all are worthy of close perusal.

Although the first reference to the quartic surface, treated generally dates no farther back than 1848, being Poncelet's article in the seventh volume of the first series of the *Nouvelles Annales de Mathématiques*, the study of quadric surfaces gave early rise to certain surfaces of the fourth order whose theory is closely related to that of quadrics, the normopolar surface and the first pedal of the ellipsoid (which is, in fact, none other than the surface of elasticity) being illustrations. Chief among such surfaces and the one most in-

teresting, both to the mathematician and to the physicist, is the wave surface. Though one cannot say that this surface was distinctly mentioned by Fresnel prior to his memoir read before the French Academy in 1827, it is, nevertheless, true that it was hinted at in three previous papers of his, the first appearing in 1815; I have, therefore, considered myself justified in heading the list of references to articles upon the wave surface with references to these three papers. This surface has appealed distinctly to English investigators, fully one-half of the 84 articles upon the wave surface being from the pens of English mathematicians and physicists. This list includes Hamilton, Maccullagh, Lloyd, Lubbock, Sylvester, Archibald Smith, Moon, Cayley, Weddle, Walton, Prescott, W. Roberts, Tait, Turnbull, Niven, Townsend, Hicks, S. Roberts, Heavyside and Larmor. Among the long list of articles upon the wave surface, the length of which list may be considered remarkable when one considers that a single quartic surface is being treated, the fourteen papers of Mannheim, extending from 1867–1885 are to be especially noted. It would appear to be a most meritorious piece of work to compile the results embraced in this long list of papers, representing the investigations of three-quarters of a century, into a full and concise treatise upon the wave surface.

The recent appearance of the second edition of Darboux's memoir on cycliques and cyclides, a very able and interesting review of which, by Mr. Roberts, was published in the last May number of the BULLETIN, shows the maintenance of deep interest in the anallagmatic surfaces of the fourth order. In the third volume of the *Proceedings of the London Mathematical Society*, page 189, the late Professor Cayley says that the Cyclide was first considered by Dupin in his "Applications de Géométrie, etc.," Paris, 1821, as the envelope of a sphere touching three given spheres; but as early as 1809, Dupin, in an article "Mémoire sur la sphère tangente à trois ou à quatre autres," published in the *Correspondance sur l'École Polytechnique*, hinted at this surface; and articles by Hachette, Binet, Gergonne and Poncelet, published between the years 1814 and 1820, upon the same subject, might be mentioned in the same connection. The chief honor, however, due for the present theory of anallagmatic surfaces of the fourth order, must be divided between Darboux and Casey, each of whom appears to have labored and achieved independently of the other.

In this bibliography, I have deemed it expedient, in order to avoid confusion, to separate the torus, which can be gen-

erally defined as the surface generated by the revolution of a conic around a fixed axis anywhere situated, from the cyclide proper, and also to treat the remaining anallagmatic surfaces of the fourth order separately. Bordoni's article in the eighth volume of the *Giornale di Fisica, Chimica e Storia Naturale* is the earliest paper that I find upon the torus proper.

Still another surface separates itself from the mass of quartic surfaces as possessing unusual interest, both intrinsically and historically, viz., Kummer's surface. This surface, being the 16-nodal quartic, gave immediate rise to the problem of the determination of the formula for the expression of the maximum number of nodes possessed by an n-thic surface, a problem that is, as far as I know, yet unsolved; the wave surface is, in fact, a particular case of Kummer's surface, in which but four of the nodes are real. Kummer's surface also bears an interesting relation to the theory of group transformations.

Steiner's surface, first determined by him as the Hessian of the general cubic surface, and the surface of elasticity, which I have already noted as being the first pedal of the ellipsoid, have each received enough attention to entitle them to be placed in a separate class.

Altogether, section 11 contains 390 references.

Section 12, treating of surfaces of the fifth and higher orders, need not be touched upon, except to mention that a nearly complete bibliography of quintic surfaces is given in a foot-note to the memoir, "On Quintic Surfaces," published in the first volume of *The Mathematical Review* and that the surfaces of this section, in so far as they have yet been studied, are mostly ruled surfaces, special developables, surfaces representable upon a plane and studied from that standpoint, and surfaces whose theory is inseparably connected with the theory of quadric surfaces. Among such last may be mentioned the parabolic ring of order six; the elliptic ring of order eight; the parallel surface of the paraboloid and the first central negative pedal of the ellipsoid of order ten; and the centro-surface of the ellipsoid and the parallel surface of the ellipsoid, of order twelve; all of which surfaces are remarkable in that their reciprocals are octadic 8-nodal quartic surfaces of the same species as the general torus.

The first reference to articles upon surfaces of revolution, other than quadrics, is that to an anonymous contribution to the seventh volume of the *Annales de Mathématiques pures et appliquées*, entitled "Caractères des surfaces de révolu-

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The first reference to articles upon surfaces of revolution, other than quadrics, is that to an anonymous contribution to the seventh volume of the *Annales de Mathématiques pures et appliquées*, entitled "Caractères des surfaces de révolu-

tion, * * *," 1816. To this section there are sixty-seven references which, for the most part, treat of special surfaces. Articles by De la Gournerie, in the twentieth volume of the *Journal de l'École Polytechnique* and by Codazzi in the first volume of the *Annali di Matematica* are to be especially mentioned among the earlier contributions, while the papers of Pirondini (*Nouvelles Annales de Mathématiques*, 1888; *Annali di Matematica*, 1890 and 1894) may be noted from among those of the last few years. There are some seventy articles listed in this section.

Euler's article in the 7th volume of the *Mémoires de l'Académie des Sciences de St. Pétersbourg*, 1820, is the subject of the first reference in section 14. Lamé's note and memoir in the *Journal de Mathématiques pures et appliquées*, 1838 and 1843 and Bertrand's paper in the *Journal de l'École Polytechnique*, 1843 follow. The subject of orthogonal surfaces is so closely connected with the consideration of isothermal surfaces that most articles treating of the one deal also with the other. The papers of Lamé and Bertrand, mentioned above, are, of the earlier papers, to be especially mentioned. There are 82 references to this section.

Of the 27 articles referred to in section 15 (envelopes), Livet's, in the first volume of the *Correspondance sur l'École Polytechnique*, 1803, followed closely by Monge's article in the fifth volume of the *Journal de l'École Polytechnique*, 1806, is the first to which reference is made. The envelopes treated are nearly all envelopes of planes or of spheres.

Of the 16 articles, references to which comprise the section upon reciprocal surfaces and which is introduced by reference to Salmon's paper in the 2d volume of the *Cambridge and Dublin Mathematical Journal*, the several contributions of Salmon and Cayley, together with the notable paper of Zeuthen in the 10th volume of the *Mathematische Annalen*, are the worthiest.

References to articles upon the important subject of minimal surfaces (section 17), number 90. This section is introduced by reference to Gergonne's "Considerations preliminaires," published in the 7th volume of the *Annales de Mathématiques pures et appliquées*, 1816. Among the articles referred to, those by Schwarz (extending from 1865 to date), Lie and Enneper deserve a special mention. A minimal surface of peculiar interest is Enneper's celebrated surface of the ninth order (cf. *Zeitschrift für Mathematik und Physik*, volume 9). An interesting historical sketch and rapid review of minimal surfaces by Dr. Hancock is now running through *The Mathematical Review*.

To articles upon ruled surfaces in general there are ninety references, the first being to Binet's paper in the second volume of the *Correspondance sur l'École Polytechnique*, 1809. The first strictly epoch-making paper, however, is that of Chasles, in the eleventh volume of the *Correspondance mathématique et physique* (*Quetelet*), 1839, pp. 49–113, entitled "Sur les surfaces engendrés par un ligne droite." There is, however, an important paper of date earlier than this, viz., 1826, that has been, as far as I know, overlooked. I refer to the paper of Karal Hube, "Dalcy ciag, etc." (On the formation of curved surfaces by a right line), published in the eleventh volume of the *Rocznik Towarzystwa Nankowegoz Uniwersystetem Krakowskim Polaczonego.*

Olivier's three articles in the *Bulletin des Sciences de la Société Philomathique de Paris*, 1832–1833 and the *Extraits des Procés-Verbaux des Séances de la Société Philomathique*, 1838, are the first articles that I find treating specifically of scrolls. Of the fifty articles referred to in this division of section 18, the papers by Cayley, "On Skew Surfaces, otherwise Scrolls" (*Cambridge and Dublin Mathematical Journal*, volume 7 and *Philosophical Transactions of the Royal Society of London*, volumes 153, 154 and 159) are to be noted not only as being the first English contributions, but as epoch-making in themselves.

Of torses in general (not noting special developables), Poisson's articles, "Sur les surfaces developables," published in the *Bulletin des Sciences de la Société Philomathique de Paris*, 1825, and in the first volume of Baumgartner's *Zeitschrift*, 1826, are the earliest I find. Of the 60 articles listed in this division, many treat of special developables.

Jellett's paper, "Sur la surface dont la courbure moyenne est constante," in the *Journal de Mathématiques pures et appliquées*, 1853, is the first of the 30 papers referred to as treating specifically of surfaces of constant curvature. Lie's papers in the fourth and fifth volumes of the *Tidsschrift for Mathematik* are the ones to be especially noted.

Of the many special studies of geometry that have, from time to time, attracted the French school and to which it has attached itself, with its customary ardor, for a longer or shorter period of time, none has been more pronounced and noticeable than the study of surfaces defined by their lines of curvature. Attracted to the subject early in 1853, Bonnet had a 200-page memoir in *Journal de l'École Polytechnique* and J. A. Serret a 50-page paper in the *Journal de Mathématiques pures et appliquées* before the year was over; and, to this day, the subject has been an engrossing one to

the French school. Of English contributions, Cayley's paper in the eleventh volume of the *American Journal of Mathematics* might be mentioned.

Section 20, treating of special surfaces, contains references to all articles upon surfaces not included in previous sections. To this section, containing 340 references, space for but few remarks can be given. Applicable surfaces are introduced by Poisson's celebrated papers upon elastic surfaces, inserted in the third volume of the *Correspondance sur l'École Polytechnique*, 1812, the *Mémoires de la classe des sciences mathématiques et physiques de l'Institut*, 1812 and the *Bulletin des Sciences de la Société Philomathique de Paris*, 1814. The papers by Weingarten in the fifty-ninth volume of the *Journal für die reine und angewandte Mathematik*, the *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen* for 1887, and the one hundred and twelfth volume of the *Comptes rendus hebdomadaires*, as well as Codazzi's article in the *Mémoires présentés par divers savants*, etc., for 1883 should be also mentioned. The only English article I find is Cayley's review in the twenty-fourth volume of the *Proceedings of the London Mathematical Society*.

Of isothermal surfaces, Lamé's papers in the *Annales de Chimie*, 1833, the *Journal de Mathématiques pures et appliquées*. 1837, 1843, 1874, the *Mémoires presentés par divers savants*, etc., 1838 and Despeyron's memoir in the *Journal für die reine und angewandte Mathematik*, 1846 are to be especially remarked.

An article upon parallel surfaces dates back as far as 1813, being Bordoni's "Sopra le * * * superficie parallele" which appeared in the sixteenth volume of the *Memorie dell' Accademia di scienze, lettere ed arti in Modena*. S. Robert's articles in the fourth and fifth volumes of the *Proceedings of the London Mathematical Society*, 1873–1874, are the principal English contributions to the theory.

Normal surfaces are introduced by Brenner's paper in the thirteenth volume of the *Archiv der Mathematik und Physik*, and polar surfaces by Mainardi's article in the first volume of the *Atti dell' Istituto Lombardo di scienze, lettere ed arti*.

Of the rich literature upon pedal surfaces, there are to be especially mentioned the papers by W. Roberts in the tenth and twelfth volumes of the *Journal de Mathématiques pures et appliquées*, that of Tortolini in the thirty-first volume of the *Journal für die reine und angewandte Mathematik* and the articles by Hirst in the second volume of the *Annali di Matematica*, the third volume of *The Quarterly Journal*, the sixty-second volume of the *Journal für die reine und angewandte*

Mathematik and the twelfth volume of the *Proceedings of the Royal Society of London*. The origin of the theory goes back to Maclaurin, *Philosophical Transactions of the Royal Society of London*, 1718.

Concerning the remaining varieties of surfaces of this section space will permit no comment.

The general curve line, not lying in a single plane, was first studied by Pitot, 1728, who gave to such curves the name of curves of double curvature; afterwards by Clairaut, 1731 ("Recherches sur les courbes à double courbure,") who gave expressions for the projections of these curves, for their tangents, normals, arc, etc.; by Monge, 1771 ("Mémoire sur les developées, etc.,") published 1785, in the tenth volume of the *Mémoires présentées par divers savants*, etc., as well as in his "Applications de l'analyse à la géométrie," who gave expressions for the normal plane, center and radius of curvature, evolutes, polar lines and polar developable, center of osculating sphere, for the criterion for points of simple inflexion (where four consecutive points are in one plane) and for points of double inflexion (where three consecutive points lie upon one straight line); by Tinseau, 1774 ("Solutions de quelques problèmes, etc.," published in the ninth volume of the *Mémoires presentées par divers savants*, etc., 1780), who there considered the osculating plane* and the developable generated by the tangents; by Lacroix, in his *Calcul Differentiel*, 1790, who was the first to render the formulæ symmetrical by the introduction of the differentials of the three coördinates. The first reference of the century in this bibliography (section 21) is that to the memoir of Lancret (Mèmoire sur les courbes à double courbure") 1802, published in the first volume of the *Mémoires* * * * *savants étrangers de l'Institut*, who calculated the angle of torsion and introduced the consideration of the rectifying lines and the rectifying surfaces. For further historical detail, the reader is referred to the able and admirable memoir of St. Venant (from which most of the above facts are taken), inserted in the eighteenth volume of the *Journal de l'École Polytechnique*, 1845. Of later papers, those of W. Schell, "Allgemeine Theorie der curven doppelter Krümmung in rein geometrischen Darstellung," Leipzig, 1859, and P. Serret, "Theorie nouvelle géométrique et mécanique des lignes à double courbure," Paris, 1860 are to be especially mentioned. Enne-

* The osculating plane, however, had been previously considered as far back as 1728, by John Bernouilli in his discussion of the problem: "In superficie quacunque curva ducere lineam inter duo puncta brevissimam," (cf. his *Opera omnia*, volume 4, p. 113).

per's paper in the *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, 1881, and Valentiner's in the second volume of the *Acta Mathematica* might be also remarked. This sections contains over 300 references.

Section 22 contains 40 references and section 23 contains 87 references.

Twisted cubics appear to have been first noted by Möbius in his classic *Barycentric Calculus*, 1827, and the first paper listed in this section is Poncelet's contribution to the seventh volume of the *Correspondance mathématique et physique*, 1832. Other papers to be noted are the thirty-third note of Chasles *Aperçu historique*, his article in the *Journal de Mathématiques pures et appliquées*, 1857, Schröter's contribution to the fifty-sixth volume of the *Journal für die reine und angewandte Mathematik* (as well as his work, published by Teubner in Leipzig, 1880), and the papers of Cremona in the first, second and fifth volumes of the *Annali di Matimatica*, and in the fifty-eighth, sixtieth and sixty-third volumes of the *Journal für die reine und angewandte Mathematik;* to these should be added the articles by Sturm in the seventy-ninth and eightieth volumes of the last mentioned Journal and the fortieth volume of the *Zeitschrift für Mathematik und Physik*, 1894, the papers by G. Loria (*Rendiconti dell' Accademia della scienze, etc.*, Napoli, volume 29, 1885), by Reye (*Mitteilung der Hamburger Mathematischen Gesellschaft*, 1880), by Klein (*Annali di Matematica*, 1891) and by Valeri (*Memorie dell' Accademia di scienze, lettere ed arti in Modena*, 1892); this last paper, as well as the last mentioned paper by Sturm, deals with metric properties. The papers of Cayley in the ninth and tenth volumes of the *Journal de Mathématiques pures et appliquées*, 1844–1845, of Salmon in the forty-second volume of the *Journal für die reine und angewandte Mathematik*, 1851, of Hamilton in the eighth volume of the *Proceedings of the Irish Academy*, 1865, of Spottiswoode in the *Proceedings* and the *Philosophical Transactions of the Royal Society of London*, 1881, and of Dixon in the twenty-third and twenty-fourth volumes of *The Quarterly Journal*, 1889, are the principal English contributions. There are 111 references to section 24.

Twisted quartics of the first kind, being the intersection of two quadric surfaces, were naturally the first general species of twisted curves studied, and their history is identical with the history of quadric surfaces. The first article, reference to which is made in this section, is one by Hachette, inserted in the first volume of the *Correspondance sur l'École Polytechnique*, 1804 (Des courbes du quatrième de-

gré, etc.). Among articles upon twisted quartics of the first kind that deserve especial mention, may be noted the papers of Chasles, found in the fifty-fourth and fifty-fifth volumes of the *Comptes rendus hebdomadaires*, 1862, of Tortolini, in the fifth volume of the *Annali di Matematica*, 1863, of Painvin, in the *Nouvelles Annales de Mathématiques*, volume 7, 1868, of Reye, *Annali di Matematica*, volume 2, 1876, of Weyr, *Sitzungsberichte der math.-natur. Klasse der Akademie zu Wien*, 1876, 1877 and 1878, of Schröter, *Journal für die reine und angewandte Mathematik*, 1882 (and in his "Grundlage einer reingeometrischen Theorie der Raumcurven vierter Ordnung erster Species," Leipzig, B. G. Teubner, 1890), of Lange, *Zeitschrift für Mathematik und Physik*, 1883 and of Milanowski, *Journal für die reine und angewandte Mathematik*, 1884.

Twisted quartics of the second kind were first shown to exist by Salmon, 1849, in his paper, "On the classification of curves of double curvature," inserted in the fifth volume of the *Cambridge and Dublin Mathematical Journal*, in which it was shown that every quartic of this kind always could be obtained as the residual intersection of a quadric surface and a cubic cone having one of the generators of the quadric for a double edge. Steiner independently arrived at this same fact in 1856 in his paper "Ueber die Flächen dritten Grades" published in the fifty-third volume of the *Journal für die reine und angewandte Mathematik* and in the *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, 1856, Cremona's paper in the *Rendiconti delle sessioni dell' Accademia delle scienze dell'Istituto di Bologna* and the *Annali di Matematica* for 1861, Bertini's, in the *Reale Istituto Lombardo di scienze i lettere, Rendiconti*, 1872, Eberhardt's, in the *Zeitschrift für Mathematik und Physik* 1877, Stahl's, in the *Journal für die reine und angewandte Mathematik*, volume 101 and Rohn's, in the forty-second and forty-third volumes of the *Berichte über die Verhandlungen der Sächsischen Gesellschaft zu Leipzig* are all papers worthy of particular mention. Altogether section 25 contains 106 references.

Of articles upon twisted quintics, those by Bertini ("In Memoriam Dominici Chelini, Collectanea Mathematica." Höpli, 1881), by Weyr (*Sitzungsberichte der math.-natur. Klasse der Academie zu Wien*, volumes 90, 92 and 97), by Montesano (*Rendiconti dell' Accademia delle scienze*, etc., Napoli, 1888) and by Berzolari (*Memorie del l'Accademia dei Lincei*, 1893), are the principal.

The papers of Weyr (*Comptes rendus hebdomadaires*, volume 76 and *Sitzungsberichte der math.-natur. Klasse der Academie zu Wien*, volumes 99 and 100) and the paper of London

(*Mathematische Annalen*, volume 44), are the principal contributions to the theory of the twisted sextic; with the exception of the first, which deals with the classification of sextic curves in space, these all treat of the twisted sextic of deficiency one.

Of papers upon twisted curves of orders higher than six, Weyr's article in the sixty-ninth volume of the *Sitzungsberichte der math. natur. Klasse der Academie zu Wien* ("Ueber Raumcurven siebenter Ordnung"), is the only one in which a general treatment is found. Section 26 contains, altogether, twenty-two references.

Section 27 (special forms of twisted curves), containing 140 references, embraces the list of all articles, references to which could not find a fit place in the preceding sections.

To the observant student, such a bibliography as the one here briefly sketched, is interesting in many ways other than those at first apparent. It shows the trend of the different schools and their exoduses from one system or line of thought to another; the swarming of disciples after masters as the latter have changed the directions or methods of their researches; the fluctuating enthusiasm in progressive fields of investigation; the numerous accidental duplications of discoveries and demonstrations, especially between different schools and languages; and, most important of all, it lays bare the "rag ends" that the masters, grown tired of one field of investigation, have left convenient for the disciple to lay hold of and unravel. Finally, it is believed, that to know well that which has already been done best fits one to do that which is, as yet, undone.

COLUMBIA UNIVERSITY.

LINEAR DIFFERENTIAL EQUATIONS.

Handbuch der Theorie der linearen Differentialgleichungen. Von Professor Dr. LUDWIG SCHLESINGER, Privatdocenten an der Universität zu Berlin. In zwei Bänden. Erster Band. Leipzig, Teubner, 1895. 8vo, xx+486 pp.

The literature of the theory of linear differential equations has increased so rapidly during the last quarter of a century, and much of it has become of such importance that a detailed and methodical presentation of the subject such as is contained in the volume before us will be welcomed by large numbers of mathematicians and advanced students. While the author has attempted to make the

book as accessible as possible to different classes of readers by developing as far as he needs them various subjects not coming directly under the heading of differential equations, the book is in no sense an introduction to the theory. It is a work to be referred to by persons who already know the subject in outline at least and wish detailed information in some special direction, not to be worked through chapter by chapter.

The first thing which strikes us on opening the volume is the arrangement of the table of contents, which covers thirteen pages and for which the contents of Lacroix's *Calculus* has served as a model. The interesting feature here is the extensive list of references (to original memoirs, etc.), which is given in connection with each section. This will explain in part the remarkable freedom of the pages of the book from foot notes as these references are not repeated. The advantages of this method are obvious, but it also has the disadvantage that it is impossible to make the precise bearing of the references as clear when they are given in this form as if they were immediately connected with the subject in hand.

The volume opens with a historical introduction due in part to Paul Günther, who was to have coöperated with Professor Schlesinger in the publication of the book, but who died shortly after the work was undertaken. The greater part of this introduction is devoted to an account of the historical development of the theory of analytic functions and of the general theory of differential equations, only the last three pages treating of the case of *linear* equations. Here and in a few historical statements scattered through the book we find an error in historical perspective which is so often met with elsewhere that it seems necessary for us to refer to it.

The founder of the modern theory of linear differential equations was Riemann. Although his paper of 1857 on the hypergeometric series refers to a very special class of differential equations only, the methods there developed were the ones which, according to Riemann's suggestion, have been used in the further development of the theory. These methods, like most others due to Riemann, were not generally understood at the time. In 1865, however, Fuchs, inspired by the lectures of Weierstrass,* as well as by Rie-

*It is here stated (cf. *Crelle* Vol. 66, p. 122; see also a statement by Thomé on p. 322 of the same volume) that the theorem concerning the development of solutions about non-singular points is reproduced from Weierstrass's lectures. This proof is explicitly ascribed by Schlesinger to Fuchs.

mann's paper, treated in an easily intelligible manner two important special problems. It was not, however, in the memoir of 1857 alone that Riemann laid the foundation of the modern theory of linear differential equations. In a posthumous paper on minimum surfaces published in 1867, very important geometrical methods involving linear differential equations of the second order were introduced which, as developed by Schwarz,* form the basis of the theory of automorphic functions. However much Fuchs may have done in his numerous contributions towards the development of the theory of linear differential equations, he is in no way its founder as he is termed by Schlesinger and by many others.

After the historical introduction comes a nine page *Introduction*, in which some matters concerning differential equations in general and the singular points of analytic functions are explained.

Besides these introductions the volume before us consists of eight parts (Abschnitte), each of which is subdivided into chapters and these again into sections with headings.

In Part I., which consists of only fourteen pages, Weierstrass's proof of the fundamental existence theorem for linear differential equations is first given† and the analytic extension of the elements of the function thus obtained is considered, the general case in which the coefficients of the equation are multiple valued functions being at once taken up. Fundamental systems and the linear dependence of solutions are then treated.

Part II. is devoted to the more formal sides of the theory which bring out the analogy of linear differential equations to algebraic equations. Among the questions considered are the determination of the common solutions of two differential equations; the breaking up of homogeneous linear expressions into symbolic factors; the reduction of the order of a homogeneous linear differential equation when one of its solutions is known; the theory of multipliers and ad-

* "Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt." *Crelle* Vol. 75, 1872. The methods referred to had been in part independently discovered by Schwarz in 1867; cf. Schwarz's collected works, Vol. I., p. 109.

† An error which occurs on p. 123 of Fuchs's memoir is here copied. There will be no constants $M_k > |p_k(x)|$ throughout a circle surrounding a non-singular point and reaching out to the nearest singular point. It is necessary to draw a second circle concentric with the first and a little smaller, as is done in a similar case on p. 165, where Frobenius serves as the model.

joint differential equations; the fundamental theorems concerning non-homogeneous differential equations. Finally a brief discussion of the conception of reducibility as applied to linear differential equations is given.

Part III. refers to the behavior of the solutions of the differential equation in a doubly connected region within which the coefficients are single valued. The necessary theorems concerning the composition of linear substitutions are first developed, no specific knowledge except of the theory of determinants being assumed. The fundamental equation is defined and discussed from various points of view, and is finally applied to the development of solutions of the differential equation about points in the neighborhood of which the coefficients of the equation have the character of algebraic functions.

In Part IV. the theory of regular singular points (Punkte der Bestimmtheit) is considered.* Assuming that the coefficients of the equation behave like rational functions in the neighborhood of the point in question, a necessary form for these coefficients at a regular point is obtained, and it is then shown that this form is also sufficient to ensure the regularity of the point. There follows a detailed study of the solutions of the equation in the neighborhood of a regular point. Finally the results obtained are extended in two directions, first to the case of non-homogeneous equations and secondly to the case in which the coefficients of the homogeneous equation have the character of algebraic functions in the neighborhood of the point in question.

Part V. is entitled: "Differential equations of the Fuchsian class," *i. e.*, equations which are everywhere regular. It is first shown that the more general case in which the coefficients are algebraic functions can be reduced to the case in which they are rational functions, and this case alone is then considered. The results of part IV. are applied to the equations in question and are extended by the consideration of the question of the convergence of the series used on their circles of convergence. The general form of the differential equation which is everywhere regular and for which the singular points and their exponents are given is then obtained, and after certain simple special cases, including the equation with constant coefficients; have been treated, the hypergeometric differential equation is taken up and, after being reduced to the unsymmetrical

*Concerning the terminology here used, cf. the first foot note on p. 89 of this volume of the BULLETIN.

form in which Gauss used it, treated at some length. Although rather a matter of form than of substance, it should be mentioned that neither here nor elsewhere is it clearly brought out that the point at infinity does not play a part essentially different in the theory of linear differential equations from that played by any other point. This makes a symmetrical development of the theory of the hypergeometric function quite impossible.

Part VI. treats of the representation of solutions in a region which is bounded by two concentric circles and within which the coefficients are single valued analytic functions, including the important special case of the development about an irregular point. The method of infinite determinants is first taken up,* and then a method of Hamburger depending upon conformal transformation. Each of these methods while giving a complete solution of the problem involves a determination of constants by transcendental processes. The other two methods which are here treated apply in special cases only and even then usually give merely particular solutions. They involve, however, only algebraic determinations of constants. The first of these methods refers to the case in which some of the solutions are regular at an irregular point while the second concerns Thomé's "normal solutions."

While the series which have so far been used to represent the solutions of differential equations converge within regions bounded by one circle or by two concentric circles only, the question is taken up in Part VII. of obtaining solutions by means of infinite processes which converge throughout the whole plane, the case of equations with rational coefficients being alone explicitly considered. Caqué's method as generalized by Fuchs is first explained and a few applications of it are given. The rest of this part is devoted to the discussion of the neighborhood of an irregular point for equations with rational coefficients, Poincaré's discussion of the semi-convergent developments in the neighborhood of such points being, however, barely touched upon.

The eighth and last part is again concerned with equations with rational coefficients, the problem here being to give methods for the computation of the coefficients of the

* Concerning the mathematical connection between Hill's original use of these determinants and the use to which they are here put cf. a paper by Burkhardt: Ueber einige mathematische Resultate neuerer astronomischer Untersuchungen, insbesondere über irreguläre Integrale linearer Differentialgleichungen. Papers of the Chicago Mathematical Congress, pp. 20–25.

substitutions which express a fundamental system of solutions in the neighborhood of one point in terms of a fundamental system in the neighborhood of another point when we pass from one point to the other by a given path. Several methods are explained and the part closes with an application to the differential equation satisfied by the hypergeometric series.

The foregoing brief summary of the contents of the volume conveys no idea of the mass of detail given. This has not, especially in the later parts, been carried to the extent of including all known results concerning the subjects treated. The amount of detail given is, however, still very great, so great in fact as to obscure at times the underlying ideas, and for this reason, we repeat, the book will be found most useful to those who already have a clear grasp of the leading principles of the subject.

At various points in the book several more or less essentially distinct methods of treatment were open to the writer. In such cases it is to be regretted that Professor Schlessinger should often have been satisfied with the exposition of one of these methods even though, as is usually the case, it is the best. Advanced students for whom the volume before us is written, refer to a handbook not merely to find out what the facts in a certain branch of the subject are and how they can be established, but what their relations with one another and with other subjects are, and each new method of treatment suggests new relations. A few examples will make clear what we mean.

The fundamental existence theorem is established in Part I., by means of Weierstrass's proof of convergence of the development into a power series. This proof is simple and historically interesting. Frobenius's proof of convergence when applied to the special case of non-singular points is, however, still simpler. Even more striking than this is the proof of this existence theorem which can be based upon Caqué's solution given in Part VII. The possibility of this method is not even hinted at nor is any reference given.* Another matter is the necessary and sufficient condition that a set of functions should be linearly independent. This condition is deduced at the beginning of Part II. from the results of Part I., but it is not pointed out that this theorem can be proved directly and can then serve as a foundation for much of the work of Part I., making it in particular unnecessary to prove directly that the circle of convergence of

*See for references p. 53 of this volume of the BULLETIN.

the developments into power series about a non singular point reaches out to the nearest singular point of the differential equation. A last example of the same kind concerns the development of solutions both about regular and about irregular points in the cases in which logarithms enter. The method originally used by Fuchs in these cases is not even referred to in the text.* In all these cases and in many more of the same sort a brief explanation of the general outlines of these alternative methods, together with references for the details, would have been sufficient.

Somewhat in line with the question just considered, but involving far more important matters is the treatment accorded to Riemann's theory of the hypergeometric function. The differential equation which this function satisfies is treated as an important example of the general theory and the chief results of Riemann and his predecessors are obtained, but no attempt is made to convey to the reader anything of the spirit of Riemann's remarkable memoir. This is the more to be regretted as the methods used by Riemann have not, as has sometimes been suggested, been superseded by more modern methods; on the contrary the general theory of linear differential equations has not as yet grown up to the point where all of Riemann's methods can be generally applied.

We have confined ourselves so far to the criticism of details and have refrained from any general discussion of the scope, arrangement, and completeness of the work as a whole. This course has been made necessary by the fact that only the first of the two volumes of which the Handbook is to consist is as yet before us. We cannot, however, refrain from referring to one matter of a general character and that a very important one, the entire omission of the theory of the real solutions of real differential equations. We refer here not merely to the proofs of the existence of solutions when the coefficients of the equations are not assumed to be analytic,† but also to the important class of

* A modification of Frobenius's method for the treatment of regular points has recently been published by Kneser (*Math. Ann.*, Vol. 47, p. 408). The object of this modification is to avoid the use of Weierstrass's theorem that a uniformly convergent series of analytic functions may be differentiated term by term. It is not as well known as it should be that while the proof of this theorem as given by Weierstrass is rather complicated, its proof by means of Cauchy's integral is immediate (cf. for instance Demartres: Cours d'Analyse, Vol. 2, p. 74). There can, therefore, be no object in avoiding the use of this theorem except for the purist who wishes to base everything with Weierstrass on the use of power series.

† For references see pages 53 and 54 of the present volume of the BULLETIN.

theorems represented by Sturm's famous paper in the first volume of *Liouville's Journal*. When we consider the great importance of these questions both theoretically and in physical applications it is hard to justify their omission in what claims to be a Handbook of the whole theory of linear differential equations. This omission may, perhaps, be in part made good in the second volume.

Although Professor Schlesinger's treatise fails to meet some of the demands which it seems to us may fairly be made of a Handbook, it is certain, owing to the great amount of information which it contains in accessible form, to fill an important place in every mathematical library.

MAXIME BÔCHER.

HARVARD UNIVERSITY,
December, 1896.

TABLE OF THE FIRST FORTY ROOTS OF THE BESSEL EQUATION $J_0(x) = 0$ WITH THE CORRESPONDING VALUES OF $J_1(x)$.

Presented to the American Mathematical Society at its Third Summer Meeting, September 1, 1896.

BY R. W. WILLSON AND B. O. PEIRCE.

THE first ten values of x for which Bessel's function of the zeroth order, $J_0(x)$, vanishes have been given to ten places of decimals by Meissel.* The next thirty roots of the equation, $J_0(x) = 0$ and the values of $J_1(x)$ corresponding to these forty roots have been computed by us by means of Vega's ten place table of logarithms† except in the few cases where a greater number of places was necessary, and for these we have had recourse to Thoman's tables.‡ All the values have been checked by duplicate computation and the first four values of $J_1(x)$ by comparison with Meissel's tables.

* *Math. Abhandlungen der K. Akad. der Wissenschaften zu Berlin*, 1888.
† Thesaurus Logarithmorum Completus, Lipsiae, 1794.
‡ Tables de Logarithmes à 27 Décimales pour les Calculs de Précision, Paris, 1867.

The p^{th} root in order of magnitude of the equation $J_0(x) = 0$ is denoted by x_p.

p	x_p	Log x_p	$J_1(x_p)$	Log $(\pm J_1(x_p))$.
1	2.4048255577	0.3810835788	+0.51914750	9.7152908
2	5.5200781103	0.7419452231	−0.34026481	9.5318170
3	8.6537279129	0.9372032361	+0.27145230	9.4336935
4	11.7915344391	1.0715703238	−0.23245983	9.3663479
5	14.9309177086	1.1740865018	+0.20654642	9.3150177
6	18.0710639679	1.2569837232	−0.18772880	9.2735309
7	21.2116366299	1.3265741787	+0.17326589	9.2387131
8	24.3524715308	1.3865430443	−0.16170155	9.2087142
9	27.4934791320	1.4392297006	+0.15218121	9.1823610
10	30.6346064684	1.4862123057	−0.14416598	9.1588628
11	33.7758202136	1.5286059043	+0.13729694	9.1376609
12	36.9170983537	1.5672275586	−0.13132463	9.1183462
13	40.0584257646	1.6026938781	+0.12606950	9.1006100
14	43.1997917132	1.6354816528	−0.12139863	9.0842138
15	46.3411883717	1.6659671666	+0.11721120	9.0689691
16	49.4826098974	1.6944525978	−0.11342918	9.0547248
17	52.6240518411	1.7211842839	+0.10999114$_6$	9.0413577
18	55.7655107550	1.7463656842	−0.10684789	9.0287659
19	58.9069839261	1.7701667872	+0.10395957	9.0168645
20	62.0484691902	1.7927310714	−0.10129350	9.0055816

The p^{th} root in order of magnitude of the equation $J_0(x) = 0$ is denoted by x_p.

p	x_p	Log x_p	$J_1(x_p)$	Log $(\pm J_1(x_p))$.
21	65.1899648002	1.8141807465	+0.09882255	8.9948561
22	68.3314693299	1.8346207594	−0.09652404	8.9846355
23	71.4729816036	1.8541418997	+0.09437879	8.9748744
24	74.6145006437	1.8728232368	−0.09237051	8.9655333
25	77.7560256304	1.8907340543	+0.09048519	8.9565775
26	80.8975558711	1.9079354006	−0.08871080	8.9479765
27	84.0390907769	1.9244813451	+0.08703686	8.9397032
28	87.1806298436	1.9404200023	−0.08545424	8.9317336
29	90.3221726372	1.9557943757	+0.08395493	8.9240462
30	93.4637187819	1.9706430570	−0.08253186	8.9166216
31	96.6052679510	1.9850008094	+0.08117879	8.9094426
32	99.7468198587	1.9988990584	−0.07989015	8.9024933
33	102.8883742542	2.0123663047	+0.07866100	8.8957595
34	106.0299309165	2.0254284784	−0.07748689	8.8892282
35	109.1714896498	2.0381625681	+0.07635913	8.8828610
36	112.3130502805	2.0504302219	−0.07528823	8.8767271
37	115.4546126537	2.0624112882	+0.07425684	8.8707365
38	118.5961766309	2.0740706879	−0.07326670	8.8649067
39	121.7377420880	2.0854252422	+0.07231515	8.8592293
40	124.8793089132	2.0964904866	−0.07139973	8.8536966

NOTES ON THE THEORY OF BILINEAR FORMS.

BY PROFESSOR HENRY TABER.

[Read at the November Meeting of the Society, 1896.]

ASSOCIATED with any linear substitution

$$x_r = \sum_1^n{}_s a_{rs} y_s \qquad (r=1, 2, \cdots u)$$

is the bilinear form

$$\sum_1^n{}_r \sum_1^n{}_s a_{rs} x_s u_r,$$

which we may regard as representing the substitution; and in the symbolic system described below we need not distinguish between a linear substitution and the bilinear form associated with it. Thus we may use the same symbol A to denote indifferently the substitution and the bilinear form.*

If B denotes the bilinear form $\sum_1^n{}_r \sum_1^n{}_s b_{rs} x_s u_r$, $A \pm B$ denotes either the bilinear form $\sum_1^n{}_r \sum_1^n{}_s (a_{rs} \pm b_{rs})\, x_r u_r$, or the linear substitution to which this form corresponds. Further, in this symbolic notation, $A\,B$, which is termed the product of A into B, signifies either the bilinear form $\sum_1^n{}_r \sum_1^n{}_s c_{rs} x_s u_r$, where

$$c_{rs} = \sum_1^n{}_t a_{rt} b_{ts} \qquad (r, s = 1, 2, \cdots n),$$

or its associated linear substitution, which is obtained by the composition in a definite order of the substitutions A and B. The composition or "multiplication" of bilinear

* In the notation and nomenclature invented by Cayley a linear substitution and a bilinear form are each represented by the square array of its coefficients, the *matrix* of the bilinear form or of the linear substitution; and we do not distinguish in general between a bilinear form and its matrix, or between a linear substitution and its matrix. See "Memoir on the Theory of Matrices," also "Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function," *Philosophical Transactions*, 1858. Cayley's symbolic notation differs only in unessential features from the system here described, which was invented by Frobenius subsequent to Cayley's investigations.

forms is then associative and distributive, but not in general commutative.

In this notation, the equation $A = B$ is to be regarded as satisfied identically, that is to say, for all values of the x's and u's, and is thus equivalent to n^2 equations between the corresponding coefficients of A and B. In what follows I shall denote the identical substitution, or its associated bilinear form $\sum_1^n x_r u_r$, by I. The determinant of the linear substitution or bilinear form A will be denoted by $|A|$. If $|A| \neq 0$, there is a definite form, the *reciprocal* of A, denoted by A^{-1}, such that $A A^{-1} = A^{-1} A = I$.

§ 1.

Relation between the exponents of the elementary divisors of $|A - \rho B|$ *and the numbers belonging to the roots of the equation* $|A - \rho B| = 0$.

Let ρ be a variable parameter not contained in the coefficients of A and B, the host, $A - \rho B$, of bilinear forms is then a sheaf of bilinear forms, and the determinant $|A - \rho B|$ of this sheaf is a rational integral function of ρ of order n.* Let ρ_0 be a root of the equation $|A - \rho B| = 0$; then, if $a - \rho_0 b = 0$, $a - \rho b$ is a divisor of the determinant $|A - \rho B|$. If ρ_0 is a root of multiplicity l of the above equation, $|A - \rho B|$ contains $a - \rho b$ to the power l, but to no higher power. The minors of the determinant $|A - \rho B|$ of all orders are also rational polynominals in ρ; and all the minors of order $n - r$ may contain $a - \rho b$ if $l > r$. Let l_r be the highest power of $a - \rho b$ contained in all the minors of order $n - r$. If l_ν is the last of the series l_1, l_2, etc., that is not zero, we have

$$l > l_1 > l_2 > \ldots > l_\nu > 0;$$

and, if we put

$$e = l - l_1,\ e_1 = l_1 - l_2,\ \cdots\ e_{\nu-1} = l_{\nu-1} - l_\nu,\ e_\nu = l_\nu,$$

then

$$e \geqq e_1 \geqq e_2 \geqq \ldots \geqq e_\nu \geqq 1,$$

$$(a - \rho b)^l = (a - \rho b)^e (a - \rho b)^{e_1} (a - \rho b)^{e_2} \ldots (a - \rho b)^{e_\nu}.$$

The several factors $(a - \rho b)^e$, $(a - \rho b)^{e_1}$, etc., of $(a - \rho b)^l$ are *elementary divisors* (*elementar theiler*) of the determinant $|A - \rho B|$. These elementary divisors all vanish for $\rho = \rho_0$.

* It is assumed that $|B| \neq 0$.

Corresponding to each root of the equation $|A - \rho B| = 0$ is a system of elementary divisors which all vanish if ρ be equal to the root in question. The theory of the elementary divisors of $|A - \rho B|$ is due to Weierstrass.

Associated with the roots of the equation $|A - \rho B| = 0$, are certain other numbers which have been studied by Sylvester, Buchheim, Ed. Weyr, and others. These numbers play an important rôle in the theory of linear substitution, and are closely related to the exponents of the elementary factors. Thus, ρ_0 being a root of multiplicity l of the equation $|A - \rho B| = 0$, the nullity* of the bilinear form $A - \rho_0 B$ is at least one, and cannot exceed l; and the nullity of successive powers of $A - \rho_0 B$ increases until a power of exponent $\mu \leqq l$ is attained whose nullity is equal to l. The nullity of the $(\mu + 1)^{\text{th}}$ and higher powers of $A - \rho_0 B$ is then also l; and if

$$m_1,\ m_2,\ \cdots\ m_{\mu-1},\ m_\mu = l,$$

denote respectively the nullities of

$$(A - \rho_0 B)\ \ (A - \rho_0 B)^2,\ \cdots\ (A - \rho_0 B)^{\mu-1},\ (A - \rho_0 B)^\mu,$$

we have

$$m_1 \geqq m_2 - m_1 \geqq \cdots \geqq m_\mu - m_{\mu-1} \geqq 1.$$

I term the numbers m_1, m_2, etc., the numbers *belonging* to the root ρ_0 of the equation $|A - \rho B| = 0$.

Let now a diagram be formed corresponding to ρ_0 by arranging $m_\mu = l$ dots in rows and columns so that there shall be μ rows, respectively, of

$$m_1\ ,\ m_2 - m_1\ ,\ \cdots\ m_\mu - m_{\mu-1}\ ,$$

equidistant dots, and so that the first dot in each row falls in the same (left hand) column. The number of columns will then be m_1, which will be found to be equal to $\nu + 1$; and the number of dots in the successive columns, beginning with the extreme left hand column, will be equal, severally and respectively, to $e, e_1, e_2, \cdots e_\nu$.†

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* The nullity of a bilinear form is m if the $(m-1)^{\text{th}}$ minors (the minors of order $n - m + 1$) of its matrix are all zero, but not all the m^{th} minors (the minors of order $n - m$).

† *E. g.*, if the numbers belonging to ρ_0 are 5, 9, 13, 16, the diagram corresponding to ρ_0 is

* * * * *
* * * *
* * * *
* * *

and we have, $e = 4$, $e_1 = 4$, $e_2 = 4$, $e_3 = 3$, $e_4 = 1$.

Thus knowing the numbers belonging to the root ρ_0 of the equation $|A-\rho B|=0$, we can find the exponents of the elementary divisors of $|A-\rho B|$ that vanish for $\rho=\rho_0$, and conversely. Further, from the existance of certain relations between the numbers belonging to ρ_0, we may show the existance of corresponding relations between the exponents e, e_1, etc. For example, if $\mu=1$, in which case $m_1=l$, the e's are all equal to unity, that is the elementary divisors that vanish for $\rho=\rho_0$ are linear (or *simple*); and conversely. Again if the numbers belonging to ρ_0 are all even, ν is odd, and

$$e=e_1, e_2=e_3, e_4=e_5, \cdots e_{\nu-1}=e_\nu;$$

that is, the elementary divisors $(\rho_0-\rho)^{e_\nu}$ occur in pair with equal exponent. Conversely, if this condition is satisfied, the numbers belonging to ρ_0 are all even.

If B is the identical substitution, or the bilinear form $I=\sum_1^n x_r u_r$, the determinant $|A-\rho I|$ is the *characteristic function* of A, and the equation $|A-\rho I|=0$, the *characteristic equation* of A.

§ 2. *Special linear homogeneous group.*

Let $|A|=+1$, the linear substitution A is then a transformation of the special linear homogeneous group. Let the roots of the characteristic equation of A, $|A-\rho I|=0$, be

$$\rho'=R'e^{\theta'\sqrt{-1}}, \quad \rho''=R''e^{\theta''\sqrt{-1}}, \text{ etc.},$$

respectively, of multiplicity m', m'', etc.,— R', R'', etc., being the moduli, and θ', θ'', etc., the smallest positive arguments of ρ', ρ'', etc. Then since $|A|=+1$,

$$(R')^{m'}(R'')^{m''}\cdots=+1.$$
$$m'\theta'+m''\theta''+\cdots=2k\pi,$$

where k is some positive integer. Let now δ be the greatest common divisor of the exponents of all the elementary divisors of the characteristic function of A, then by virtue of what was shown in § 1, it follows from a theorem given in the BULLETIN for April, 1896, page 232, that *A can be generated by the repetition of an infinitesimal transformation of the special linear homogeneous groap if, and only if, k contains δ.*

§ 3.

Group of linear transformations whose invariant is a bilinear form.

1. Let G denote the group of linear transformations whose invariant is the form A of non-zero determinant, the x's and u's being cogredient. Let $\breve{B}$ denote the bilinear form $\sum_1^n{}_r \sum_1^n{}_s b_{sr} x_s u_r$ conjugate (or transverse) to $B = \sum_1^n{}_r \sum_1^n{}_s b_{rs} x_s u_r$. Then, if the linear substitution associated with the form B is a transformation of group G,

$$\breve{B} A B = A,$$

and conversely. From what was stated in the introduction, it is evident that this identity gives rise to n^2 equations in the coefficients of A and B which are quadratic in the coefficients of B. Consequently, the problem to determine the coefficients of the automorphic linear transformation of a bilinear form presents itself at the outset as the solution of a system of n^2 quadratic equations in n^2 quantities (the coefficients of B). The reduction of the general solution of this problem to the solution of a system of n^2 equation linear in n^2 variables was effected by Cayley in 1858. Thus Cayley showed that if the linear substitution B is a transformation of group G, we may in general put

$$B = (A + C)^{-1}(A - C),$$

where C is a bilinear form satisfying the equation

$$(a) \qquad A^{-1}C + (\breve{A}^{-1})\,\breve{C} = 0.*$$

As shown by Cayley, this equation is satisfied identically if A is symmetric, provided C is alternate; or if A is alternate, provided C is symmetric.† A form A is, of course, symmetric if $\breve{A} = A$ (*i. e.*, if $a_{sr} = a_{rs}$), and is alternate if $\breve{A} = -A$ (*i. e.*, if $a_{sr} = -a_{rs}$).

The next step in Cayley's investigation was, if A is neither symmetric nor alternate, to express the coefficients of C (and hence of B) rationally as functions of the smallest

* *Philosophical Transactions*, 1858, p. 44.

† These two theorems are almost invariably ascribed by German mathematicians to Frobenius, by whom they were indeed given in 1878 (*Crelle*, vol. 84); but they had been given by Cayley twenty years earlier in the *Philosophical Transactions*.

number of parameters; but, as pointed out by Voss, Cayley's solution of this latter problem is erroneous. The determination of the bilinear form C satisfiying equation (α) is due to Voss.

If -1 is a root of the characteristic equation of B, this linear substitution cannot be given Cayley's representation.

In the *Mathematische Annalen*, vol. 48, p. 102, Mr. Loewy states that if A is a bilinear form (of non-zero determinant), whose automorphic linear transformations of determinant $+1$* form an irreducible system,† every proper transformation of group G results from the composition of two of Cayley's forms, that is, can be given the representation

$$(\beta) \qquad (A + C_1)^{-1}(A - C_1)(A + C_2)^{-1}(A - C_2),$$

where C_1 and C_2 satisfy equation (α).

I find that for certain forms A (of non-zero determinant), for which $|A - \breve{A}| = 0$, not every automorphic transformation of determinant $+1$ can be given this representation. But, if A is any bilinear form whatever of (non-zero determinant), such that neither the determinant $|A + \breve{A}|$ nor the determinant $|A - \breve{A}|$ is zero, every automorphic linear transformation can be given Mr. Loewy's representation, that is, results from the composition of two of Cayley's forms.

In this case, that is, if $|A \pm \breve{A}| \neq 0$, every transformation of group G is given by the second power of Cayley's expression, that is, may be given the representation

$$(\gamma) \qquad [(A + C)^{-1}(A - C)]^2,$$

where C satisfies equation (α). For, in a paper read before the Society, August, 1895, (to appear in the *Quarterly Journal*), I showed that, if neither $|A + \breve{A}|$ nor $|A - \breve{A}|$ is zero, every transformation of group G is the second power of a transformation of this group;‡ and it is readily proved that the second power of a transformation of group G is given by the square of Cayley's expression.

*If B is a transformation of group G, $|B|^2 = 1$. If A is alternate, all the transformations of group G are proper, that is are of determinant $+1$.

†See Voss. Abhandlungen der k. bayer. Akadamie der Wiss., II. Cl., XVII. Bd., II. Abth., 1890.

‡See BULLETIN, for October, 1895, p. 5. I also showed in this paper that, if A is alternate, every orthogonal substitution of group G is the second power of an orthogonal substitution of this group, and can therefore be generated by the repetition of an infinitesimal orthogonal substitution of group G.

2. Every transformation of group G that can be given by Cayley's representation may be generated by the repetition of an infinitesimal transformation of group G. *Afortiori*, every transformation of group G that can be given the representation (γ), and, therefore, the second power of any transformation of group G, can be generated thus.* It is, therefore, of some importance to determine the conditions necessary and sufficient that a transformation can be expressed by formula (γ).

For the case in which A is symmetric, I have shown in the *Proceedings of the London Mathematical Society*, vol. 26, p. 364, *et seq.*, that in order that a transformation B of group G may be the second power of a transformation of this group, it is necessary and sufficient that either -1 shall not be a root of the characteristic equation of B, or that the numbers belonging to -1 shall all be even. By what was shown in § 1 it follows that this theorem is equivalent to the following: If A is symmetric, a transformation B of group G is the second power of a transformation of this group if, and only if, the elementary factors of the characteristic equation of B that vanish for $\rho = -1$ occur in pairs with equal exponent; provided such factors exist, otherwise no condition is necessary. In this form the preceeding theorem was given by Mr. Loewy, subsequent to my article, in the number of *Annalen* above referred to.

For the case in which A is alternate, I showed in vol. 46 of the *Mathematische Annalen*, p. 582, that if -1 is a root of the characteristic equation of the transformation B of group G, the numbers belonging to -1 are all even. In the paper above referred to, read before the August meeting of the Society, 1895, I showed that these conditions were also sufficient.† This theorem was communicated to the *American Academy of Arts and Sciences*, January, 1896.‡

Therefore, by § 1, if A is alternate, the necessary and sufficient conditions that a transformation B of group G shall be the second power of a transformation of this group are that either -1 shall not be a root of the characteristic equation of B, or, if -1 is a root of the characteristic equation of B, that the elementary factors of the determinant $|B - \rho I|$ that vanish for $\rho = -1$ shall occur in pairs with equal exponent.

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*See *Proceedings American Academy of Arts and Sciences*, vol. 30, p. 181; also paper on "The Group whose Invariant is a Bilinear Form," to appear in a forthcoming number of the *Mathematical Review*.

†See BULLETIN for October, 1895, p. 5.

‡See *Proceedings*, vol. 31, p. 349.

Since, if $| A \pm \breve{A} | \neq 0$, each transformation of group G is the second power of a transformation of this group, it follows, in this case, that if -1 is a root of the characteristic equation of a transformation of this group, the numbers belonging to -1 are all even.*

3. Let it be assumed that $| A \pm \breve{A} | \neq 0$; and let the linear substitution B belong to the sub-group of orthogonal substitutions of group G. If -1 is not a root of the characteristic equation of B we may put

$$B = (I + C)^{-1} (I - C),$$

where C is alternate, and is commutative with A. From the theorem given in the foot-note $p.$ 161, it follows that any orthogonal substitution B of group G may be given the representation

$$[(A + C)^{-1} (A - C)]^2,$$

where C satisfies equation (α).

§ 4.

Group whose invariant is a real quadratic form.

If we put $u_1 = x_1$, $u_2 = x_2$, etc., A becomes a quadratic form. In this case we may assume that A is symmetric.

Let now G denote the group of real linear substitutions whose invariant is the quadratic form A. For $n = 4$, I have shown in the *Proceedings of the London Mathematical Society*, vol. 26, p. 375, that there are proper transformations of group G that are not the second power of any transformation of this group, and which, therefore, cannot be generated by the repetition of an infinitesimal transformation of this group, provided the roots of the characteristic equation of A are not all of the same sign. Whence, in this case, it follows, if $n > 4$, that there are proper transformation of group G not the second power of any transformation of the group, and for which, therefore, the elementary divisors that vanish for $\rho = -1$ do not all occur in pairs with equal exponent.†

On the other hand, if A is real, and if the roots of the

* The same is true, in this case, of the numbers belonging to the root $+1$ of the characteristic equation of a transformation of group G.

† Mr. Loewy states that, if A is real, and if -1 is a root of a transformation of group G, the elementary divisors $(1 + \rho)^{e_r}$ are necessarily all linear, and that therefore every proper transformation of group G can be given the representation of formula (γ).

characteristic equation of A are all of the same sign, in particular, if $A = I$, in which case G is the group of real orthogonal substitutions, every proper transformation of group G is the second power of a transformation of this group. Therefore, even when A is real, if the roots of the equation $|A - \rho I| = 0$ are not all of the same sign and $n \geqq 4$, G is not isomorphic with the group of real orthogonal substitutions.

NOTES.

THE Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York on Wednesday afternoon, December 30, at three o'clock, the President, Dr. HILL, in the chair. There were twenty-four members present. On the recommendation of the Council, the following persons, nominated at the preceding meeting, were elected to membership: Professor THOMAS WILLIAM EDMONDSON, New York University; Professor JAMES LAWSON PATTERSON, Union College, Schenectady, N. Y. Reports were presented by the Secretary, Librarian and Treasurer. The Secretary announced that the membership of the Society was 279, a net increase of 12 for the year. The number of new members admitted during the year was 22, and the number of withdrawals 10. The average attendance at the ordinary meetings during the year was 15; the attendance at the last annual meeting 25; at the summer meeting 30; at the colloquium 16; and at the Princeton meeting 34. The number of persons who had attended at least one meeting during the year was 82, an increase of 21 over the preceding year. Mr. Pfister and Dr. Schultze were appointed a committee to audit the Treasurer's accounts.

The chair appointed Dr. Ling and Mr. Pfister tellers for the annual election. Upon canvassing the ballots cast in person and by mail, they announced that the following ticket had been elected:

President,	Professor SIMON NEWCOMB,
Vice-President,	Professor R. S. WOODWARD,
Secretary,	Professor F. N. COLE,
Treasurer,	Professor HAROLD JACOBY,
Librarian,	Professor POMEROY LADUE.

Committee of Publication:

Professor THOMAS S. FISKE,
Professor ALEXANDER ZIWET,
Professor FRANK MORLEY.

Members of the Council to serve until December, 1899:

Professor ALFRED BAKER,
Dr. G. W. HILL,
Dr. EMORY MCCLINTOCK.

The following papers were read:
(1) Professor F. MORLEY: "Geometric construction of a linear covariant of the binary quintic."
(2) Professor A. S. CHESSIN: "On Foucault's pendulum."
(3) Professor E. W. BROWN: "Remarks upon a new lunar theory."

THE National Academy of Sciences held a scientific session at Columbia University, November 17 and 18, pure and applied mathematics being represented by Professor SIMON NEWCOMB, Dr. G. W. HILL, Professor ASAPH HALL, Professor H. A. ROWLAND, Professor R. S. WOODWARD and Mr. CHARLES S. PEIRCE. Mr. PEIRCE read the only strictly mathematical paper. Its title was "Mathematical Infinity."

CORNELL UNIVERSITY. The following advanced courses continue through the current year.—By Professor WAIT: Analytic geometry; Calculus.—By Professor JONES: Higher algebra and trigonometry.—By Professor MCMAHON: Fourier series, spherical harmonics and Bessel functions; Mathematical theory of sound.—By Professor TANNER: Binary quantics.—By Dr. MURRAY: Differential equations.—By Dr. HUTCHINSON: Continuous groups.—By Dr. SNYDER: Theory of functions.

In the Mathematical Club the theory of line geometry (*géométrie réglée*) is presented by the members in turn; and reports are given on current journals.

PROFESSOR OTTO HÖLDER has been called from Tübingen to a professorship of mathematics at the University of Königsberg. This makes the sixth change that has occurred at this university in the department of mathematics within the last three years. In 1893, Professors Lindemann, Hurwitz, Hilbert and Eberhard were all at Königsberg, and, in the short interval since, Professor Lindemann has been called to Munich, Professor Hurwitz to Zürich, Professor Hilbert to Göttingen, and Professor Eberhard to Halle. Professor Hurwitz was succeeded by Professor Minkowski, who was in turn called to Zürich, being now succeeded by Professor Hölder. English and American universities hardly afford examples of such rapid changes.

PROFESSOR H. POINCARÉ, who held the chair of Calculus of Probabilities and Mathematical Physics in the Paris Faculty of Sciences, has been transferred at his own request to the chair of Mathematical Astronomy and Celestial Mechanics rendered vacant by the death of Professor F. F. Tisserand. Professor Poincaré's successor is Professor V. J. BOUSSINESQ, who has hitherto held the chair of Physical and Experimental Mechanics.

THE tercentenary of the birth of DESCARTES was recently celebrated at Tours. A pilgrimage was made to the house at La Haye in which Descartes was born.

ACCORDING to *Science* a graduate scholarship in mathematics of the annual value of three hundred dollars, has been founded at Cornell University in honor of the late Professor JAMES EDWARD OLIVER.

PROFESSOR E. STUDY has been called from Bonn to a professorship of mathematics at the University of Greifswald.

DR. F. BUKA, professor of geometry at the Technical High School in Berlin, died December 4, 1896, aged forty-five years.

DR. J. A. HUGO GYLDEN, professor of astronomy at the University of Stockholm, died November 9, at the age of fifty-five years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ATMANSPACHER (O.). Die Grundlagen unserer Herrschaft über die Zahlen. Leipzig, Dürr, 1896. 8vo. 3 and 52 pp. Mk. 1.00

BENDT (F.). Katechismus der Differential- und Integralrechnung. Leipzig, Weber, 1896. 12mo. 16 and 268 pp. Mk. 3.00

BÔCHER (M.). Regular points of linear differential equations of the second order. Cambridge, Harvard University, 1896. 12mo. 24 pp. $0.50

CARNOT (L. N.). Réflexions sur la métaphysique du calcul infinitésimal. 5e édition. Paris, Gauthier-Villars, 1896. 8vo. 202 pp.

FISKE (T. S.). The length of a curved line. (*Science*, new series, Vol. 4, 1896, p. 724).

GENTRY (R.). On the forms of plane quartic curves. Dissertation presented to the faculty of Bryn Mawr College for the degree of doctor of philosophy. New York, 1896. 8vo. 73 pp., 24 plates. [To be obtained from the author, Vassar College, Poughkeepsie, N. Y.] $0.50

GREENHILL (A. G.). Differential and integral calculus, with applications. London, Macmillan, 1896. 8vo. 472 pp. 10s. 6d.

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